TRINITY COLLEGE

Semester Two Examination, 2021

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3&4

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words

Your name

Number of additional answer booklets used (if applicable):

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	48	35
Section Two: Calculator-assumed	13	13	100	90	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (90 Marks)**

This section has **thirteen** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

Let u, v and w be complex numbers, and r and θ be constants so that

$$|u| = r, \qquad \arg(u) = \theta, \qquad v = (\sqrt{3} - i)u, \qquad w = \frac{v}{1+i}.$$

Determine the following in terms of r and / or θ.

(a) $\text{Arg}(w)$. (3 marks)

Solution
$$\begin{aligned}\arg(w) &= \arg v - \arg(1+i)\\ &= \arg(\sqrt{3}-i) + \arg u - \arg(1+i)\\ &= -\frac{\pi}{6} + \theta - \frac{\pi}{4}\\ &= \theta - \frac{5\pi}{12}\end{aligned}$$
Specific behaviours
✓ indicates correct arguments for $\sqrt{3} - i$ and $1 + i$ ✓ uses difference of arguments for quotient ✓ uses sum of arguments for product and simplifies

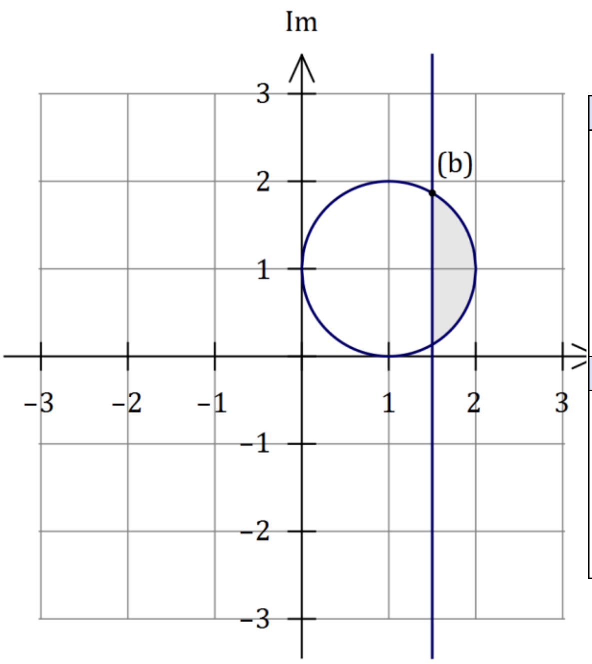
(b) $|w^2|$. (3 marks)

Solution
$$\begin{aligned}|w| &= \frac{|v|}{|1+i|}\\ &= \frac{|\sqrt{3}-1| \times |u|}{|1+i|}\\ &= \frac{2r}{\sqrt{2}}\\ &= \sqrt{2}r\end{aligned}$$ $$|w^2| = (\sqrt{2}r)^2 = 2r^2$$
Specific behaviours
✓ indicates correct arguments for $\sqrt{3} - i$ and $1 + i$ ✓ correct modulus for w ✓ correct modulus for w^2

Question 10 **(7 marks)**

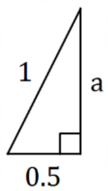
Let R be the region of the complex plane where the inequalities $|z - 1 - i| \leq 1$ and $|z + \bar{z}| \geq 3$ hold simultaneously.

(a) Sketch R on the axes below. (5 marks)



Solution
$\|z - 1 - i\| \leq 1$ is on and inside circle centre $(1, 1)$ radius 1.
$z = x + iy \Rightarrow \|z + \bar{z}\| \geq 3 \equiv \|2x\| \geq 3$ $x \leq -1.5 \cup x \geq 1.5$
Specific behaviours
✓ indicates part of R is circular arc ✓ correct radius and centre ✓ simplifies second inequality ✓ correct vertical line ✓ shades correct region

(b) Determine the maximum value of $\mathrm{Im}(z)$ in R. (2 marks)

Solution
 Circle centre to intersection: $a^2 = 1^2 - \left(\frac{1}{2}\right)^2 \Rightarrow a = \frac{\sqrt{3}}{2}$ Hence maximum value of $\mathrm{Im}(z)$ is $1 + \frac{\sqrt{3}}{2} \approx 1.866$
Specific behaviours
✓ indicates location of z ✓ correct value

Question 11 **(7 marks)**

In order to estimate the mean cost of damage sustained by parked vehicles when struck by another vehicle, an insurance company examined the records of 75 such occurrences, and obtained a sample mean of $\$3361$ with sample standard deviation $\$326$.

(a) Construct a 95% confidence interval for the mean cost of damage in all such accidents. (3 marks)

Solution
$3361 - 1.96 \times \dfrac{326}{\sqrt{75}} < \mu < 3361 + 1.96 \times \dfrac{326}{\sqrt{75}}$ $3361 - 73.8 < \mu < 3361 + 73.8$ $3287 < \mu < 3435$
Specific behaviours
✓ indicates standard deviation for sample mean ✓ indicates calculations for bounds of interval ✓ correct interval, to nearest dollar

(b) Previously, the insurance company had used the amount of $\$3290$ for the mean cost of damage in all such accidents. State, with reasons, whether this amount is no longer valid. (2 marks)

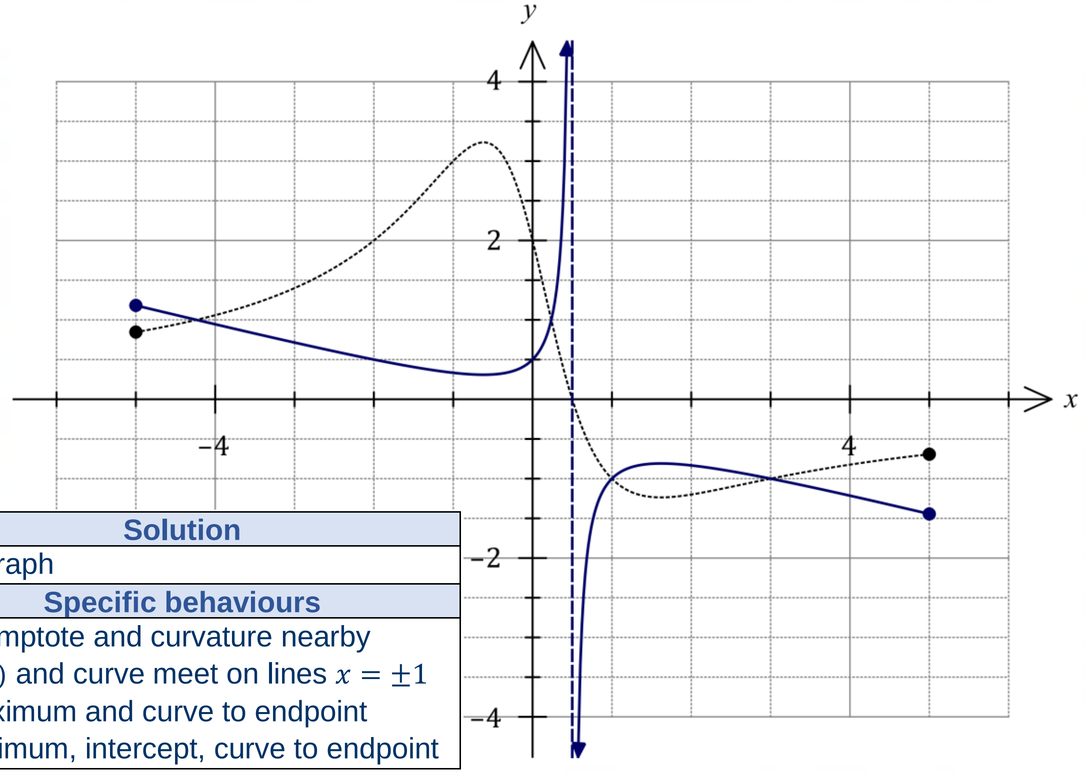
Solution
There is no reason to doubt the validity of this amount as it is contained within the bounds of the 95% confidence interval.
Specific behaviours
✓ states no reason to doubt validity ✓ states amount contained within interval

(c) State one assumption made in constructing the interval in part (a) and comment on how reasonable this assumption is in relation to the information provided. (2 marks)

Solution		
Sample means are normally distributed. Reasonable due to largish sample size of 75.	Sample was obtained randomly. Cannot comment as no information provided on how sample collected.	Sample values are independent of each other. Reasonable as such accidents are unlikely to be related.
Specific behaviours		
✓ states assumption ✓ comments on reasonableness		

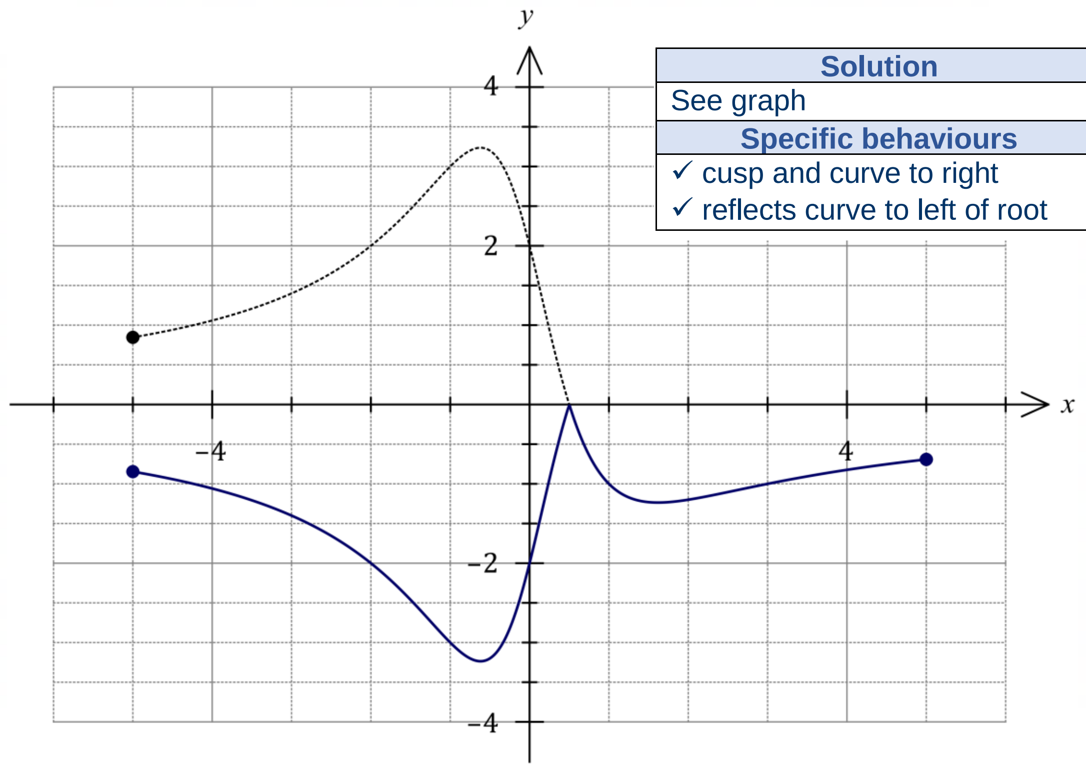
Question 12 **(6 marks)**

(a) The dotted curve on the axes below is the graph of $y = f(x)$. On the same axes, sketch the graph of $y = \dfrac{1}{f(x)}$. (4 marks)



Solution
See graph
Specific behaviours
✓ asymptote and curvature nearby
✓ $f(x)$ and curve meet on lines $x = \pm 1$
✓ maximum and curve to endpoint
✓ minimum, intercept, curve to endpoint

(b) The dotted curve on the axes below is the graph of $y = f(x)$. On the same axes, sketch the graph of $y = -|f(x)|$. (2 marks)



Solution
See graph
Specific behaviours
✓ cusp and curve to right
✓ reflects curve to left of root

Question 13 **(8 marks)**

A person who weighs 108 kg begins a specialist diet so that their rate of weight loss can be modelled by

$$\frac{dw}{dt} = k(w - 76)$$

where w is the persons weight in kilograms and t is the number of days since the diet began.

After 1 week the person had lost a total of 5.7 kg.

(a) Show use of the separation of variables method to obtain a function for w in terms of t. (5 marks)

Solution

$$\int \frac{dw}{w-76} = \int k\,dt$$
$$\ln(w-76) = kt + c$$
$$w - 76 = Ae^{kt}$$

$$t = 0, w = 108 \Rightarrow A = 108 - 76 = 32$$

$$t = 7, w = 108 - 5.7 = 102.3$$

$$102.3 = 76 + 32e^{7k} \Rightarrow k = -0.028$$

$$w = 76 + 32e^{-0.028t}$$

Specific behaviours

- ✓ separates variables
- ✓ integrates both sides, including a constant
- ✓ eliminates logs
- ✓ obtains value of constant A
- ✓ obtains value of constant k and writes function

(b) At what rate is the person losing weight after 3 weeks? (2 marks)

Solution

$$w(21) = 93.765$$

$$\dot{w} = -0.028(93.765 - 76)$$
$$= -0.498\ \text{kg/day}$$

Losing weight at a rate of 0.498 kg per day.

Specific behaviours

- ✓ weight on correct day
- ✓ correct rate of loss (allow sensible rounding)

(c) State the total weight that this person is expected to lose if they maintain the diet. (1 mark)

Solution

$$t \to \infty, w \to 76, \Delta w = 108 - 76 = 32\ \text{kg}$$

Specific behaviours

- ✓ correct weight loss

Question 14 **(8 marks)**

The coordinates of the three vertices of a triangle are $A(-2,1,3), B(-1,0,5)$ and $C(1,2,2)$.

(a) Prove that the triangle is right-angled at A. (2 marks)

Solution

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \qquad \overrightarrow{AC} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = 3 - 1 - 2 = 0$$

Hence $\angle A$ is right-angled as the scalar product of the two non-zero vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is zero.

Specific behaviours

✓ vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$
✓ forms scalar product and simplifies to 0

(b) Determine the Cartesian equation of plane that contains the triangle. (3 marks)

Solution

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$
$$= \begin{pmatrix} -1 \\ 7 \\ 4 \end{pmatrix}$$

Vector equation:

$$\mathbf{r} \cdot \mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \mathbf{n} = 21$$

Cartesian equation:

$$-x + 7y + 4z = 21$$

Specific behaviours

✓ obtains normal to plane
✓ uses point to obtain constant
✓ writes equation in Cartesian form

(c) Determine the exact vector equation of the sphere that has diameter BC. (3 marks)

Solution
Centre of sphere: $$\frac{1}{2}\left[\begin{pmatrix}-1\\0\\5\end{pmatrix}+\begin{pmatrix}1\\2\\2\end{pmatrix}\right]=\begin{pmatrix}0\\1\\3.5\end{pmatrix}$$ Radius: $$\left\|\begin{pmatrix}1\\2\\2\end{pmatrix}-\begin{pmatrix}0\\1\\3.5\end{pmatrix}\right\|=\left\|\begin{pmatrix}1\\1\\1.5\end{pmatrix}\right\|=\frac{\sqrt{17}}{2}$$ Vector equation: $$\left\|\mathbf{r}-\begin{pmatrix}0\\1\\3.5\end{pmatrix}\right\|=\frac{\sqrt{17}}{2}\ (\approx 2.062)$$

Specific behaviours
✓ calculates centre ✓ calculates radius ✓ writes equation in exact vector form

Question 15 **(7 marks)**

The mass of raisins in each 750 g packet of muesli produced by a company is normally distributed with a mean of 54.5 g and standard deviation 4.4 g.

(a) Determine the probability that the total mass of raisins in a random sample of 56 packets of muesli is at least 3080 g. (4 marks)

Solution
Let $\bar{X}$ be the sample mean of 56 packets. $\bar{X}$ is normally distributed with mean 54.5 g and standard deviation $4.4 \div \sqrt{56} \approx 0.588$ g. $\bar{x} = 3080 \div 56 = 55$ g $P(\bar{X} \geq 55) = 0.1976$
Specific behaviours
✓ states sample means normally distributed ✓ states parameters of normal distribution ✓ indicates sample mean required ✓ calculates probability

(b) Another random sample of packets is to be taken. Determine the minimum sample size required so that the chance that the sample mean mass of raisins is less than 53 g will be no more than 2%. (3 marks)

Solution
$\bar{X}$ is now distributed with mean 54.5 g and standard deviation $4.4 \div \sqrt{n}$. Require: $P\left(Z \leq \dfrac{53 - 54.5}{\frac{4.4}{\sqrt{n}}}\right) \leq 0.02$ $\dfrac{53 - 54.5}{\frac{4.4}{\sqrt{n}}} \leq -2.05375$ $n \geq \left(\dfrac{-2.05375 \times 4.4}{-1.5}\right)^2$ $n \geq 36.3$ Hence minimum sample size of 37 is required.
Specific behaviours
✓ probability statement ✓ forms inequality for n using correct z-score ✓ states minimum integer value

Question 16 **(7 marks)**

A particle travels in a straight line so that its displacement $x(t)$ cm at time t seconds, relative to fixed point O, satisfies the equation $\dfrac{d^2x}{dt^2} = -k^2x$. Initially it has a displacement of 26 cm and is moving away from O. It moves with a period of 10 seconds and an amplitude of 80 cm.

(a) Determine a suitable function for $x(t)$. (3 marks)

Solution
For SHM we can use $x(t) = A\sin(kt + \alpha)$ [or $x(t) = A\cos(kt + \beta)$]. Amplitude is $80 \Rightarrow A = 80$ and period is 10 and so $k = 2\pi \div 10 = \frac{\pi}{5}$. When $x(0) = 26 \Rightarrow \frac{26}{80} = \sin(0 + \alpha) \Rightarrow \alpha = 0.331$. $$x(t) = 80\sin\left(\frac{\pi t}{5} + 0.331\right) = 80\cos\left(\frac{\pi t}{5} - 1.234\right)$$
Specific behaviours
✓ indicates general form of solution ✓ indicates amplitude and period ✓ calculates phase shift and writes function

(b) Determine the speed of the particle when it has a displacement of 60 cm. (2 marks)

Solution
$v^2 = k^2(A^2 - x^2)$ $v^2 = \left(\frac{\pi}{5}\right)^2 (80^2 - 60^2)$ $v = \dfrac{\pi(20\sqrt{7})}{5}$ $= 4\pi\sqrt{7} \approx 33.25 \text{ cm/s}$
Specific behaviours
✓ substitutes ✓ calculates speed

(c) Determine the distance travelled by the particle in the first 5 seconds. (2 marks)

Solution
Distance travelled in one cycle is $4 \times 80 = 320$ cm. Hence in $\frac{1}{2}$ cycle will travel $\frac{1}{2} \times 320 = 160$ cm.
Specific behaviours
✓ indicates appropriate method ✓ correct distance

Question 17 **(7 marks)**

A water tank, initially empty, is in the form of an inverted right cone of radius 4 m and depth 5 m. Water is flowing into the tank at a steady rate of 0.05 m^3 per minute but leaking out at a rate of $0.002h^2$ m^3 per minute, where h is the depth of water in the tank.

(a) Determine the rate at which the depth of water is increasing in the tank when the depth of water reaches 3 m. (5 marks)

Solution
Required rate is $\frac{dh}{dt}$ when $h = 3$. Rate of change of volume of water in tank: $$\frac{dV}{dt} = 0.05 - 0.002h^2$$ Volume as function of depth using $\frac{r}{h} = \frac{4}{5}$: $$V = \frac{1}{3}\pi r^2 h$$ $$V = \frac{1}{3}\pi h\left(\frac{4h}{5}\right)^2 = \frac{16\pi h^3}{75}$$ Then $$\frac{dV}{dh} = \frac{16\pi h^2}{25} \quad \text{OR} \quad \frac{dV}{dt} = \frac{16\pi h^2}{25}\frac{dh}{dt}$$ Hence $$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{25(0.05 - 0.002h^2)}{16\pi h^2}$$ When $h = 3$ $$\frac{dh}{dt} = \frac{1}{180\pi} \approx 0.00177 \text{ m per min}$$
Specific behaviours
✓ indicates rate of change of volume wrt time ✓ uses height to radius ratio to obtain $V(h)$ ✓ expression for $\frac{dV}{dh}$ or relates $\frac{dV}{dt}$ and $\frac{dh}{dt}$ ✓ expression for $\frac{dh}{dt}$ in terms of h ✓ calculates rate, with units

(b) Explain whether the tank will ever overflow. (2 marks)

Solution
No. When $h = 5$ then $\frac{dV}{dt} = 0.05 - 0.002(5)^2 = 0$ and so the tank will be in equilibrium with flow in equal to flow out.
Specific behaviours
✓ states no, with justification ✓ explanation

Question 18 **(6 marks)**

Particle A moves with velocity vector $\mathbf{v}(t) = 4t\mathbf{i} - 2t\mathbf{j} - 3\mathbf{k}$ ms^{-1}, where t is the time in seconds and $t \geq 0$. Initially, the particle has position vector $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + 7\mathbf{k}$.

(a) Determine $\mathbf{r}(t)$, the position vector of A at time t. (2 marks)

Solution
$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \begin{pmatrix} 2t^2 + 3 \\ -t^2 + 1 \\ -3t + 7 \end{pmatrix}$
Specific behaviours
✓ integrates velocity vector ✓ uses initial condition to obtain position vector

A second particle B moves with constant velocity vector $3\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$ and has initial position vector $20\mathbf{i} + 25\mathbf{j} + 5\mathbf{k}$.

(b) Determine if the paths of the particles cross and if so, whether they meet. (4 marks)

Solution
$\mathbf{r_B}(s) = \begin{pmatrix} 20 + 3s \\ 25 - 8s \\ 5 - 2s \end{pmatrix}$ Require $\mathbf{r_A}(t) = \mathbf{r_B}(s)$ Equate i-coefficients: $2t^2 + 3 = 20 + 3s$ Equate k-coefficients: $-3t + 7 = 5 - 2s$ Solving simultaneously: $t = 4,\ \ s = 5$ Check j-coefficients. $A: -4^2 + 1 = -15$ and $B: 25 - 8(5) = -15$. Hence the paths of the particles cross but the particles do not meet. *NB. Alternative is to obtain* 3 *equations in* s, t *and solve with CAS*
Specific behaviours
✓ position vector for B at time s ✓ uses coefficients to form two equations in s and t ✓ solves equations for $s, t > 0$ ✓ checks third coefficient for consistency and interprets solution

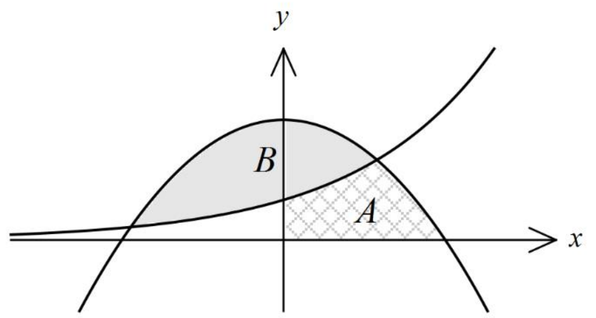
Question 19 **(7 marks)**

Let $f(x) = 4 - x^2$ and $g(x) = 3^x$.

The diagram, not to scale, shows the graphs of $y = f(x)$ and $y = g(x)$.

Region A, in the first quadrant, is bounded by the x-axis, the y-axis and the two curves.

Region B, shaded, is bounded by the two curves.



(a) Determine the area of region A. (3 marks)

Solution
$4 - x^2 = 3^x \Rightarrow x = 1, x = -1.97112$ $A = \int_0^1 g(x)\,dx + \int_1^2 f(x)\,dx$ $= \frac{2}{\ln 3} + \frac{5}{3} \approx 1.8205 + 1.6667 \approx 3.487$
Specific behaviours
✓ integrands ✓ correct limits ✓ evaluates

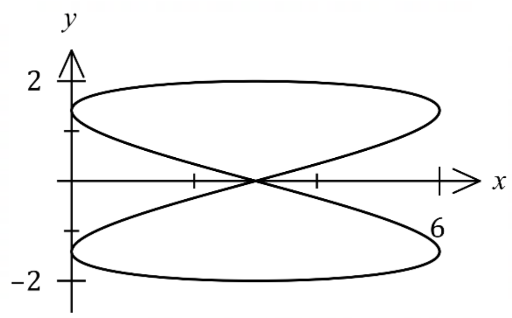
(b) Determine the volume of the solid generated when region B is rotated about the horizontal line $y = -2$. (4 marks)

Solution
Volume generated by rotating region about the line $y = -2$ will be identical to volume generated by translating both functions two units upwards and rotating about the x-axis. $V = \pi \int_{-1.97112}^{1} [(f(x) + 2)^2 - (g(x) + 2)^2]\,dx$ ≈ 163.356 NB separate integrals: $V \approx 246.545 - 83.189$
Specific behaviours
✓ adds constant to functions ✓ integrand(s) ✓ correct limits ✓ evaluates

Question 20 **(7 marks)**

The path of a particle with position vector $\mathbf{r}(t)$ is shown in the diagram, where t is the time in seconds since motion began and

$$\mathbf{r}(t) = \begin{pmatrix} 3 - 3\sin(2t) \\ 2\sin(t) \end{pmatrix} \text{ cm.}$$



(a) State the time at which the particle first touches the y-axis. (1 mark)

Solution
$3 - 3\sin(2t) = 0 \Rightarrow \sin(2t) = 1 \Rightarrow t = \frac{\pi}{4}$ s
Specific behaviours
✓ correct time

(b) Determine the Cartesian equation for the path of the particle. (3 marks)

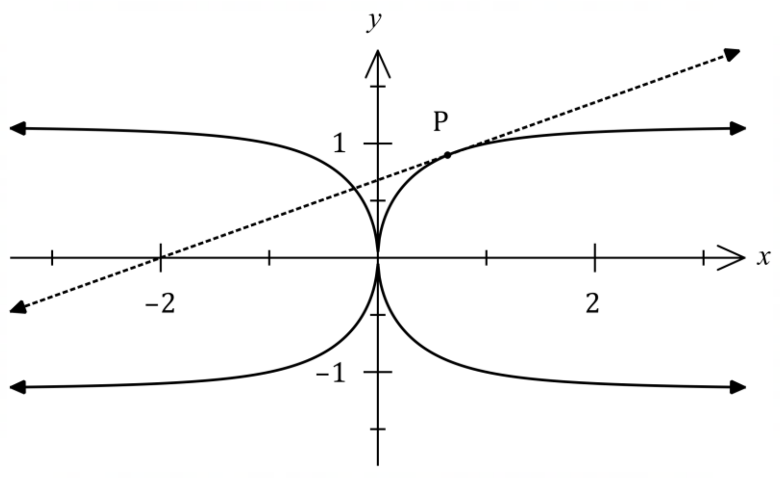
Solution
$y = 2\sin(t) \Rightarrow \sin(t) = \frac{y}{2}$ $x = 3 - 3\sin(2t)$ $\frac{3-x}{3} = 2\sin(t)\cos(t)$ $\frac{3-x}{3} = y\cos(t) \Rightarrow \cos(t) = \frac{3-x}{3y}$ $\sin^2(t) + \cos^2(t) = 1 \Rightarrow \left(\frac{y}{2}\right)^2 + \left(\frac{3-x}{3y}\right)^2 = 1 \Rightarrow (3-x)^2 = 9y^2\left(1 - \frac{y^2}{4}\right)$
Specific behaviours
✓ eliminates double angle ✓ expressions for $\sin t$ and $\cos t$ ✓ uses identity to eliminate t

(c) Determine the length of one circuit of the path. (3 marks)

Solution
Period is 2π. $v(t) = \begin{pmatrix} -6\cos(2t) \\ 2\cos(t) \end{pmatrix} \Rightarrow \lvert v(t)\rvert = \sqrt{36\cos^2(2t) + 4\cos^2(t)}$ $L = \int_0^{2\pi} \sqrt{36\cos^2(2t) + 4\cos^2(t)}\,dt$ ≈ 26.2 cm
Specific behaviours
✓ obtains velocity vector ✓ writes correct integral for length ✓ obtains length with units

Question 21 **(7 marks)**

The graph of the relationship $y^2(3x^2+y^2)=4x^2$ is shown below, together with the tangent to the curve at P that passes through the point $(-2,0)$.



(a) Use implicit differentiation to obtain an expression for $\frac{dy}{dx}$. (3 marks)

Solution
$3x^2y^2+y^4=4x^2$ $6xy^2+6x^2y\frac{dy}{dx}+4y^3\frac{dy}{dx}=8x$ $\frac{dy}{dx}=\frac{4x-3xy^2}{3x^2y+2y^3}$ *NB May expand and then diff or diff and then expand*
Specific behaviours
✓ differentiates product correctly ✓ differentiates remainder correctly ✓ rearranges for $\frac{dy}{dx}$

(b) Determine the slope of the curve at the point $(1,1)$. (1 mark)

Solution
$\frac{dy}{dx}=\frac{4-3}{3+2}=\frac{1}{5}$
Specific behaviours
✓ correct value

(c) Deduce that the x-coordinate of P is a solution to the equation $9x^3 - 2x^2 + 8x - 8 = 0$.
(3 marks)

Solution
The equation of the tangent through $(-2, 0)$ is: $y - 0 = m\big(x - (-2)\big) \Rightarrow \frac{y}{x+2} = m$ But from part (a), $m = \frac{dy}{dx}$ and so: $\frac{y}{x+2} = \frac{4x - 3xy^2}{3x^2y + 2y^3}$ ... (1) $3x^2y^2 + 2y^4 = 4x^2 + 8x - 3x^2y^2 - 6xy^2$ $6x^2y^2 + 2y^4 = 4x^2 + 8x - 6xy^2$ NB From given relationship, $3x^2y^2 + y^4 = 4x^2$, and so: $2(4x^2) = 4x^2 + 8x - 6xy^2 \Rightarrow y^2 = \frac{4-2x}{3}$ ... (2)   Substitute for y^2 in relationship: $\left(\frac{4-2x}{3}\right)\left(3x^2 + \left(\frac{4-2x}{3}\right)\right) = 4x^2$ ... (3) $2(2-x)(9x^2 + 2x - 4) = 36x^2$ $9x^3 - 2x^2 + 8x - 8 = 0$
Specific behaviours
✓ uses gradients to obtains equation (1)   ✓ obtains equation (2)   ✓ obtains equation (3) in one variable and simplifies

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________